Discovering Sum Ways of Expressing 981

34th Eastern Shore High School Mathematics Competition

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A Problem

Find digits A,…,I such that

1. ABC + DEF = GHI,

2. A through I are distinct ,

and 3. Each of 1, 2, …, 9 is used exactly once.

218 + 349 = 567

How many solutions are there to this problem?

Possible Approaches

- Count the number of possible sums
- Count the number of possible addends
- Count how many solutions there are for a given sum

$$218 + 349 = 567 \qquad 219 + 348 = 567$$
$$248 + 319 = 567 \qquad 249 + 318 = 567$$

Possible Approaches

- **Count the number of possible sums**
- Count the number of possible addends
- Count how many solutions there are for a given sum
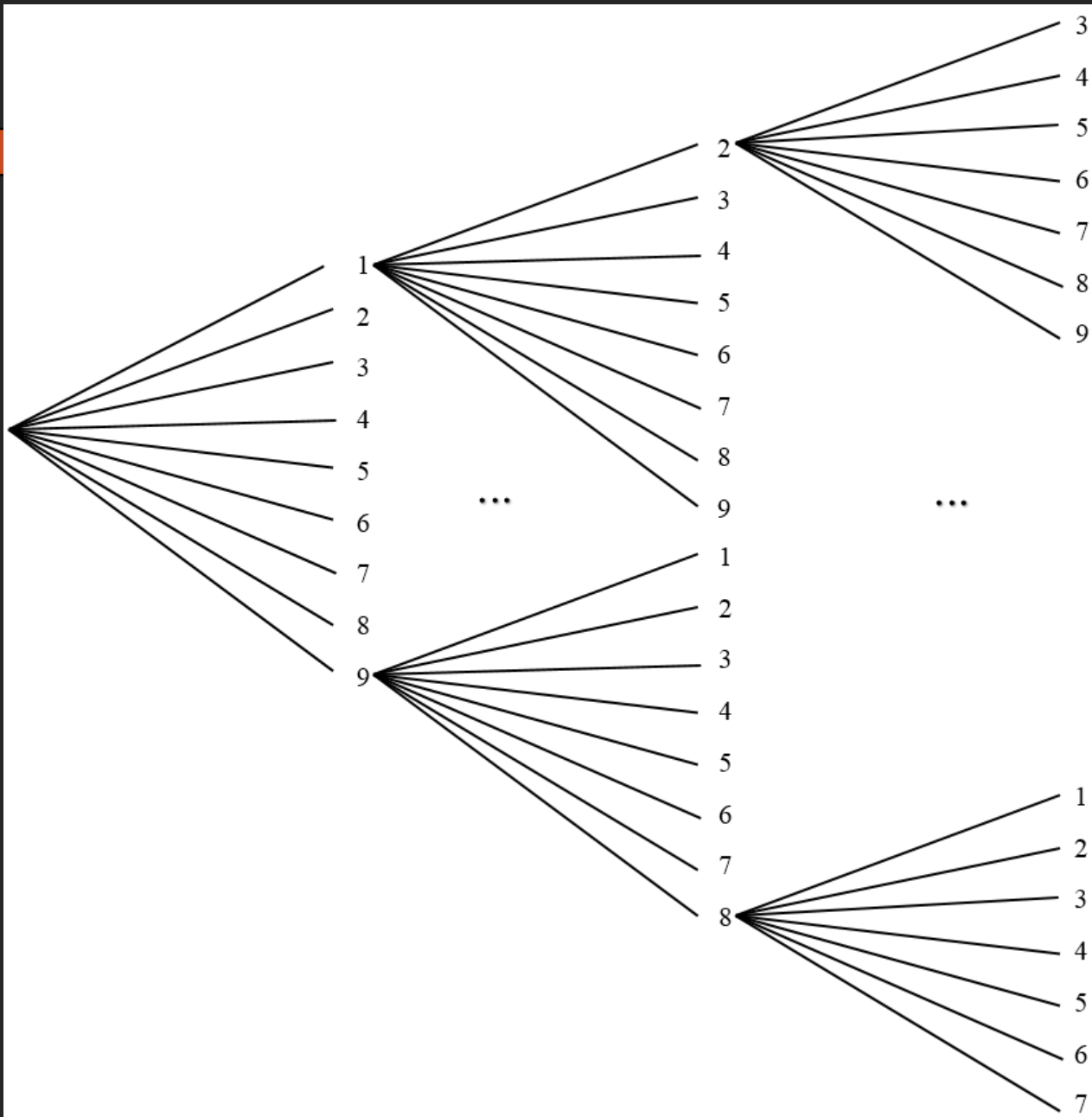
$218 + 349 = 567$ $219 + 348 = 567$

$248 + 319 = 567$ $249 + 318 = 567$

A Little Counting

$9(8)(7) = 504$ 3-digit numbers containing no 0's such that all 3 digits are distinct.

A Little Counting

$9(8)(7) = 504$ 3-digit numbers containing no 0's such that all 3 digits are distinct.

But…the first digit of the sum cannot be 1 and cannot be 2. (Why not?)

What Else Can We Say?

The first digit cannot be 3 either:

$$1BC$$
$$+2EF$$
$$\overline{3HI}$$

$B + E < 10$

So, $\{B, E\} = \{4, 5\}$ and $H = 9$.

$\{C, F, I\} = \{6, 7, 8\}$ is impossible.

What Else Can We Say?

$6(8)(7) = 336$ 3-digit numbers containing no 0's such that all 3 digits are distinct and the first digit is at least 4.

What else can we say?

Some Modular Arithmetic

We can use *modular arithmetic* to help.

Think remainders:

"7 is congruent to 1 mod 3"

means

7 ÷ 3 has a remainder of 1

Some Modular Arithmetic

We can use *modular arithmetic* to help.

Think remainders:

"8 is congruent to __2__ mod 3"

means

8 ÷ 3 has a remainder of 2

Some Modular Arithmetic

We can use *modular arithmetic* to help.

Think remainders:

"24 is congruent to $\underline{\ 0\ }$ mod 3"

means

24 ÷ 3 has a remainder of 0

Some Modular Arithmetic

We can use *modular arithmetic* to help.

Think remainders:

Any integer will be congruent to either

0, 1, or 2 mod 3.

Some Modular Arithmetic

How can we do arithmetic "mod 3?"

What is $7 + 10$ congruent to mod 3?

$$7 \equiv 1 \;(\text{mod } 3)$$

$$10 \equiv 1 \;(\text{mod } 3)$$

So, $7 + 10 \equiv 1 + 1 \;(\text{mod } 3)$.

Some Modular Arithmetic

How can we do arithmetic "mod 3?"

What is 7(10) congruent to mod 3?

$$7 \equiv 1 \ (\text{mod } 3)$$

$$10 \equiv 1 \ (\text{mod } 3)$$

So, $7(10) \equiv 1(1) \ (\text{mod } 3)$.

Back to the Problem

ABC + DEF = GHI

$$100A + 10B + C + 100D + 10E + F = 100G + 10H + I$$
$$A + B + C + D + E + F \equiv G + H + I \ (\text{mod } 3)$$

Also,
$$A + B + C + D + E + F + G + H + I = 1 + \ldots + 9 = 45 \equiv 0 \ (\text{mod } 3)$$

What could A + … + F and G + H + I be congruent to mod 3?
$$1: A + \ldots + F + G + H + I \equiv 1 + 1 \ (\text{mod } 3) \ \text{NO}$$
$$2: A + \ldots + F + G + H + I \equiv 2 + 2 \ (\text{mod } 3) \ \text{NO}$$

So, $G + H + I \equiv 0 \ (\text{mod } 3)$.

Back to the Problem

ABC + DEF = GHI

So, G + H + I ≡ 0 (mod 3).

This means that our sum GHI ≡ 0 (mod 3) also.

Remember that a whole number is divisible by three when the sum of its digits is divisible by three.

More Modular Arithmetic

$$100A + 10B + C + 100D + 10E + F = 100G + 10H + I$$

$$A - B + C + D - E + F \equiv G - H + I \ (\text{mod } 11)$$

$$A + C + D + F + H \equiv B + E + G + I \ (\text{mod } 11)$$

$$A + \ldots + I \equiv 2(B + E + G + I) \ (\text{mod } 11)$$

$$2(B + E + G + I) \equiv 1 \ (\text{mod } 11)$$

$$\text{?} \quad B + E + G + I \equiv 6 \ (\text{mod } 11)$$

More Modular Arithmetic

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

$B + E + G + I \equiv 6 \ (\mathrm{mod}\ 11)$

Look at B, E, and H:

We will have different cases based on whether we need to carry from the ones place or to the hundreds place.

More Modular Arithmetic

$$B + E + G + I \equiv 6 \ (\text{mod } 11)$$

Look at B, E, and H:

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

B, E, H relationship	Carry?	
B + E = H	None	

More Modular Arithmetic

$B + E + G + I \equiv 6 \;(\text{mod } 11)$

Look at B, E, and H:

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

B, E, H relationship	Carry?	
B + E = H	None	
B + E + 1 = H	From Ones Place Only	

More Modular Arithmetic

$B + E + G + I \equiv 6 \pmod{11}$

Look at B, E, and H:

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

B, E, H relationship	Carry?	
B + E = H	None	
B + E + 1 = H	From Ones Place Only	
B + E = H + 10	To Hundreds Place Only	

More Modular Arithmetic

$$B + E + G + I \equiv 6 \ (\text{mod } 11)$$

Look at B, E, and H:

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

B, E, H relationship	Carry?	
B + E = H	None	
B + E + 1 = H	From Ones Place Only	
B + E = H + 10	To Hundreds Place Only	
B + E + 1 = H + 10	From Ones Place and to Hundreds Place	

More Modular Arithmetic

$B + E + G + I \equiv 6 \pmod{11}$

Look at B, E, and H:

$$\frac{\begin{array}{r} ABC \\ +DEF \end{array}}{GHI}$$

B, E, H relationship	Carry?	Value of G + H + I
B + E = H	None	6 (mod 11)
B + E + 1 = H	From Ones Place Only	7 (mod 11)
B + E = H + 10	To Hundreds Place Only	7 (mod 11)
B + E + 1 = H + 10	From Ones Place and to Hundreds Place	8 (mod 11)

Back to the Problem

$$412 \leq GHI \leq 987$$

$$7 \leq G + H + I \leq 24$$

and $G + H + I$ is a multiple of 3.

9, 12, 15, 18, 21, 24

Also, $G + H + I$ must be congruent to 6, 7, or 8 mod 11.

So, $G + H + I = 18$.

Possible Sums

459   468   486   495   549   567   576

594   639   648   657   675   684   693

729   738   756   765   783   792   819

837   846   864   873   891   918   927

936   945   954   963   972   981

(Three of these don't work. Which ones?)

981

$235 + 746 = 981$

$236 + 745 = 981$

$245 + 736 = 981$

$246 + 735 = 981$

$324 + 657 = 981$

$327 + 654 = 981$

$354 + 627 = 981$

$357 + 624 = 981$

How many solutions are there in all?

What are they?

Try it!

Thank you!

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