Developing 4th Grade Students' Multiplicative Thinking

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Introduction

Many individuals recall "learning" multiplication by memorizing facts with the use of timed tests, drills, and flash cards. Educators who teach multiplication in this way try to move their students rapidly toward determining answers at the expense of helping them reason deeply about the meaning of multiplication (Kling & Bay-Williams, 2015). In doing so, students do not develop the ability to derive answers using strategic reasoning and known facts. Additionally, students who "learn" multiplication by rote memorization may not realize multiplication has multiple meanings and applications (Wallace & Gurganus, 2005).

Students who have "learned" multiplication in this way often do not retain what they have memorized and lack the ability to regenerate forgotten facts (Kling & Bay-Williams, 2015). Such students' lack of knowledge is likely to limit their ability to solve real-world word problems (Wallace & Gurganus, 2005). According to research, if students have a functional conceptualization of multiplication and have the ability to flexibly derive solutions using various strategies, they are likely to be able to regenerate a forgotten fact and use multiplication to solve problems in the real-world (Kling & Bay-Williams, 2015).

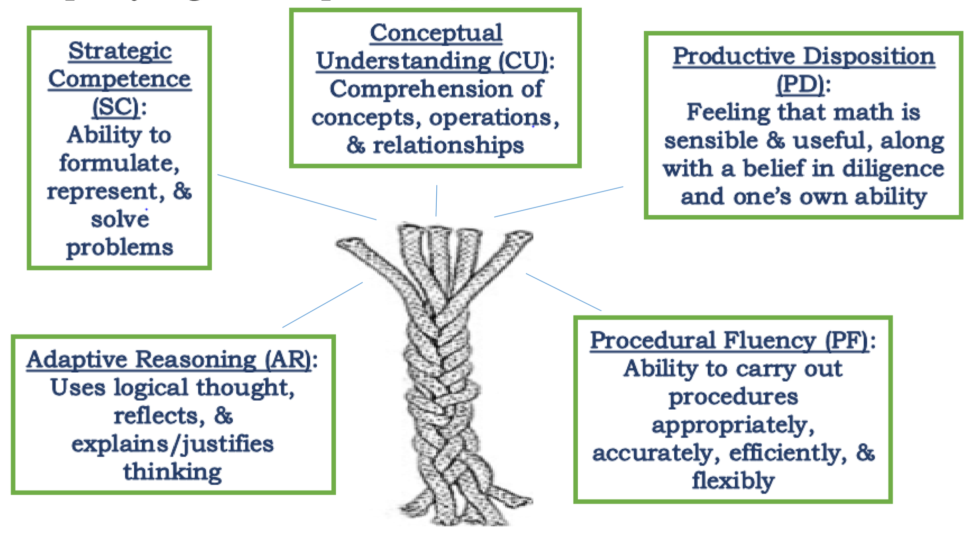
Purpose: To explore the way in which students think about multiplication and to intervene to help them develop their mathematical proficiency and a functional conceptualization of multiplication.

Research Question: How can students' mathematical proficiency be developed for multiplication?

Theoretical framework

Conceptualization of Mathematical Proficiency

Mathematical proficiency can be conceptualized by the five, interwoven strands, each of which must work together to attain proficiency (National Research Council, 2001). The National Research Council (2001) formulated the following depiction of the strands and the accompanying descriptions (p.116-117).



Ideas from Literature Integrated In Our Study

"Three Steps to Mastering Multiplication Facts"
(Kling & Bay-Williams, 2015)

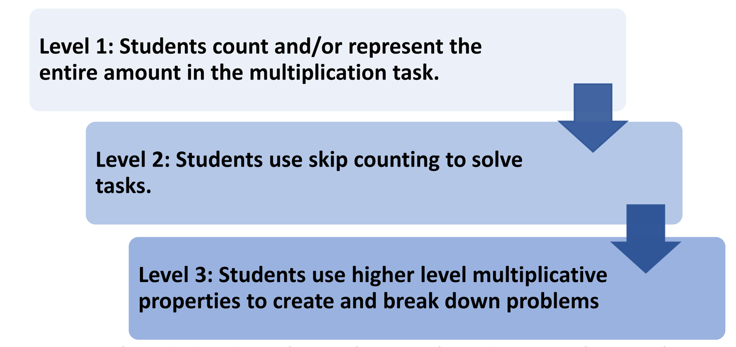
- ❖ Students must progress through three phases to master multiplication: model/count to find answer, derive answer using reasoning strategies/known facts, and mastery
- ❖ Encourage strategic thinking and using known facts
- ❖ Use area model to encourage decomposing factors

"Conceptualizing Division with Remainders"
(Lamberg & Wiest, 2012)

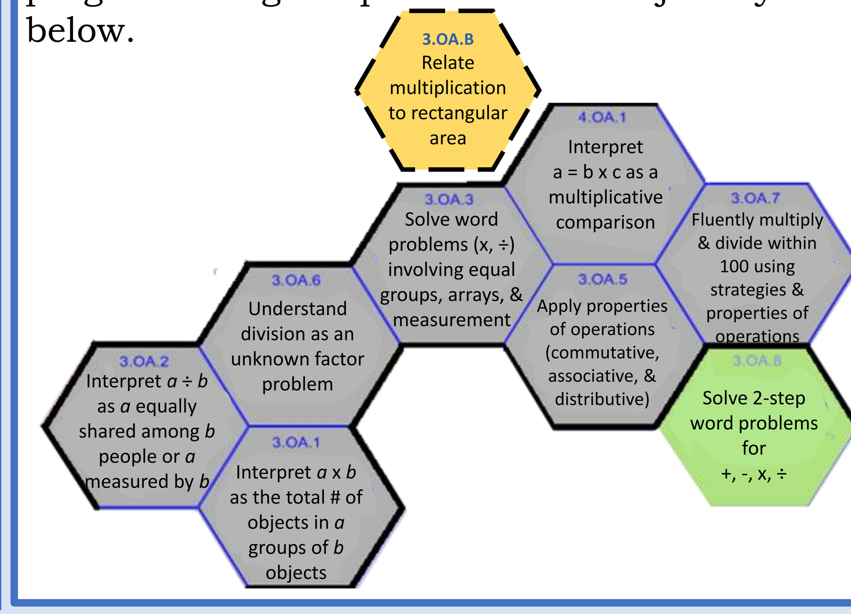
- ❖ Have students create/solve their own story problems
  - requires them to think about the meaning of the operations they want the "solver" to use
- ❖ Physically manipulating objects helps students understand meanings of operations (division)

CCSS Learning Progression for Multiplication

The Common Core State Standards (CCSS) Writing Team (2011) outlined how a student's learning progresses with regard to multiplication. Students' solutions and multiplicative development can be categorized into the following three levels (p.25-26).



Confrey et al. (2012) developed 18 student learning trajectories for the Common Core State Standards for Mathematics (National Governor's Association for Best Practices & Council of Chief State School Officers, 2010). One of the trajectories was the Multiplication and Division trajectory. Our goal was to help students progress along the portion of the trajectory shown below.



Methodology

Participants and Procedure

Participants/Students

# of participants: 4

Genders: 2 girls, 2 boys

Pseudonyms for students: Elliott, Megan, Piper, Trevor

Grade Level: Incoming 4th grade students

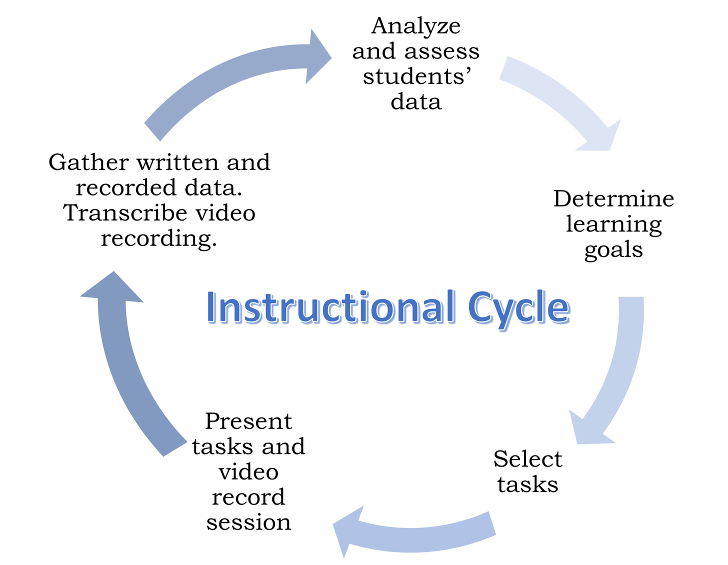
Participation Rate: 100% attendance for all 4 students

Sessions:

- Pre-assessment (30 minute clinical interview for each student)
- Seven 1-hour instructional sessions
- Post-assessment (30 minute clinical interview for each student)

Procedure

Each week, we used the following procedure in our effort to help each student make progress along the trajectory and towards mathematical proficiency in multiplication.



Methodology

Data gathering and analysis

Pre and Post Interview: Key Tasks

A pan of brownies that is twelve inches in one direction and two inches in the other is cut into one-inch square pieces.
Draw a picture/use a manipulative to show pan of brownies once it has been cut.
Without counting 1-by-1, how many brownies are there?

On a school field trip, 72 students will be traveling in 9 vans. Each van will hold an equal number of students. The equation shows a way to determine the number of students that will be in each van. 72 ÷ 9 = ?
This equation can be rewritten using a different operation. Place the operation and number pieces we will provide you in the proper boxes.
☐ ☐ ☐ · 72

Suppose there are 4 tanks and 3 fish in each tank. The total number of fish in this situation can e expressed as 4x3=12.
a. What is meant in this situation by 12 ÷ 3 = 4?
b. What is meant in this situation by 12 ÷ 4 = 3?

Data Gathering

- Video recorded and transcribed all sessions
- Collected and archived all written work from students

Analysis

- Coded transcripts using five strands of proficiency
- Summarized strengths, weaknesses, & made conjectures about how to address weaknesses during next session
- Evaluated students' progress with respect to trajectory

References

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Confrey, J., Nguyen, K.H., Lee, K., Panorkou, N., Corley, A.K., & Maloney, A.P. (2012). *TurnOnCCMath.net: Learning trajectories for the K-8 Common Core Math Standards*. Retrieved from https://www.turnonccmath.net

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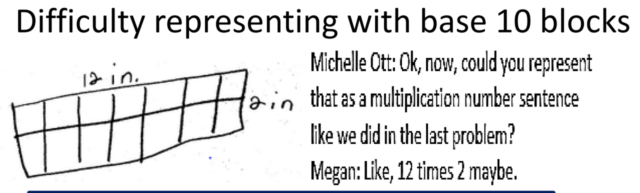
Pre-Assessment

All students had attained Level 1 with regards to their multiplicative development and some exhibited Level 2 and 3 reasoning. When determining the product, Piper had to count 1-by-1, while others recalled their multiplication facts or fluently used skip counting to determine the solution. The automatic fact recall did not appear to be consistently accompanied by CU as students had difficulty making the connection between the computed product and the visual representation.

Array Model: All students struggled with the brownie problem (see above).

- Had to count 1-by-1 to solve
- AR weaknesses (e.g. struggled to apply reasoning used to draw a picture to construct a model and vice versa)
- Difficulty representing with base 10 blocks



Michelle Ott: Ok, now, could you represent that as a multiplication number sentence like we did in the last problem?
Megan: Like, 12 times 2 maybe.

Megan demonstrated SC, PF, and CU weaknesses. Using the picture, she said there were 14 brownies. Using the base 10 blocks, she said there were 24.

Conceptualizing of Division: One of the most striking observations was Piper's difficulties with division. She said she was familiar with division, but she had difficulty with each strand. 72 ÷ 9 = 72

School Field Trip problem (see above): Piper arranged the pieces as depicted above. She could not explain why she did this. She demonstrated SC, PF, SC, and AR weakness with regard to division.

Michelle Ott: Let's see what everyone wrote. Now, everyone show your boards to each other. This is what we're going to be doing, today. Whenever we ask a question, you'll answer it on the whiteboard. Then, you're going to share it with everyone in the group. (Reads Megan's board) 12 divided by 6 equals 2. Piper?

Piper: (shrugs shoulders)

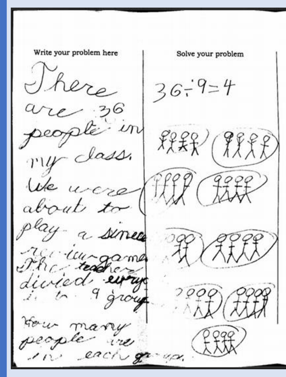
MO: You didn't get a number sentence?

Piper: No (shakes head).

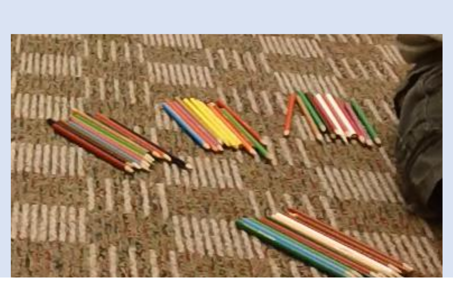
MO: Now, you have 12 eggs and we want to divide it into 2 groups.

Piper: (no response)

Lesson 2 Work Sample: Piper demonstrated SC, PF, and CU weaknesses. She was the only student who had extreme difficulty representing the problems symbolically.



Lesson 3 Work Sample: Megan demonstrated CU of division. She wrote a "sharing" division word problem.



Trevor: – I was thinking like somebody's name has 44 pencils and wants to figure out if he can make 11 groups.

Lesson 3 Work Sample: Trevor demonstrated an AR weakness. He is very procedurally fluent, but fails to reflect on his number selection. His arrangement does not match his context.

Empirical Teaching and Learning Trajectory

3.OA.3: Solve x and ÷ word problems involving equal groups, arrays, & measurement quantities

Lesson 1: Egg Cartons & Multiplication

Students had difficulty with array model during interview.
To begin our study of multiplication, we used ideas from an NCTM *Illuminations* lesson (Willman, 2015).
- Students designed rectangular egg cartons to hold 24 eggs
- Wrote a multiplication number sentence for each carton.

Lesson 2: Egg Cartons & Division

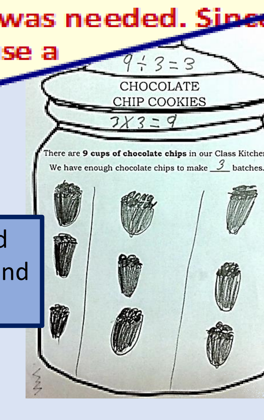
We wanted to provide students with the opportunity to relate the cartons to division and develop CU of the relationship between multiplication and division.
- Students given story problems describing cartons with X number of rows. Egg packaging company had to fit exactly 24 eggs in each carton (24 eggs divided into X rows)
- Selected and shared multiplication and division number sentences for each carton
-Discussed and reasoned to determine who was correct

Lesson 3: Create Your Own Division Story Problems

During lesson 2, Piper had difficulty with each strand of proficiency. Individualization needed. We will revisit arrays after more understanding has been attained. Students needed work with various division contexts, not just arrays. Lamberg and Weist (2012) emphasized the importance of having students physically divide objects and using the objects to write story problems for various contexts.
- Each student given two bags of manipulatives & used objects to write ÷ word problems
- Solved and checked their work using objects
- Partner solved problem

Lesson 4: How many batches of cookies can we make?

Megan, Piper, and Trevor showed CU of division. They accurately wrote problems involving "sharing." Trevor lacked the ability to stop and reflect (AR). Elliott lacked CU. He had difficulty describing a context. More exposure to division contexts was needed. Since none of the students wrote "measurement" problems, we decided to use a context that measuring naturally occurs – baking cookies.
- Each student was assigned an ingredient and determined how many batches they could make using the amount of each ingredient needed for 1 batch and amount in the "Class Kitchen"

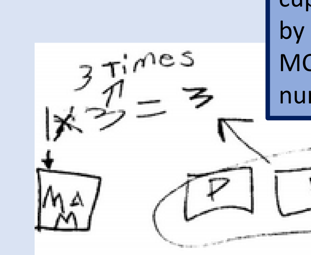


Lesson 4 Work Sample: Piper demonstrated PF, SC, and CU by accurately representing and solving the problem.

4.OA.1: Interpret multiplicative comparisons (MC)

Lesson 5: Using MCs to Make Trail Mix

Students were successful during lesson 4. Piper showed PF, SC, and CU of division. Elliott interpreted context and used division appropriately. Students were ready to learn new application of multiplication-MC It was important that we used comparisons that were natural from a multiplicative stance (Nunes & Bryant, 1999).
- Interpreted and used MCs to make trail mixes using manipulatives



Lesson 5 Work Sample: One recipe called for "three times as many pretzels as M&Ms (1 cup)." Megan demonstrated PF by accurately interpreting the MC and determining the correct number of cups of pretzels.

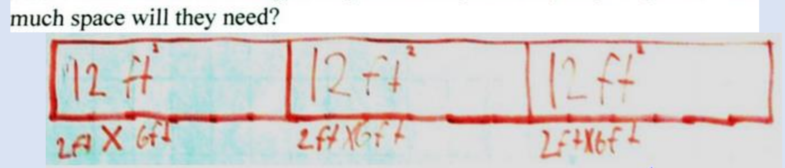
3.OA.B: Relate multiplication to rectangular area

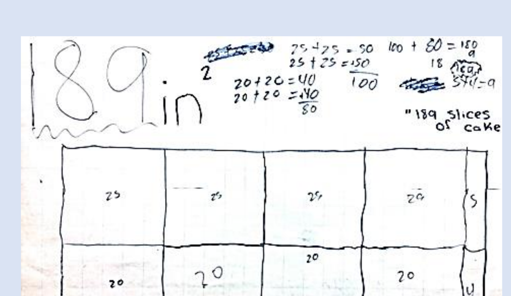
Lesson 6: Area & Gardens

Students showed CU and PF.
They were ready to readdress arrays and incorporate area. We used Kling and Bay-Williams' (2015) ideas to relate multiplication to area and help them develop ability to decompose factors and use distributive property.
- Found area of gardens by decomposing 1 factor

Lesson 6 Work Sample: Piper decomposed 18 and applied the distributive property to determine the area.

Farmer Eli wants to plant a pumpkin patch. Each pumpkin needs a 1 foot wide by 1 foot long space to grow. Farmer Phoebe thinks the patch should be 2 feet wide and 18 feet long. If they decide to plant the pumpkin patch, how much space will they need?





Lesson 7 Work Sample: A 9''x 21'' sheet cake was cut into 1 square inch slices. Elliott decomposed 9 and 21, solved the partial products, and determined the area and number of slices for the entire cake by adding the partial products. He demonstrated SC, PF, and CU.

CAPSTONE EXPERIENCE
3.OA.8: Solve 2-step word problems with + , - , x , ÷

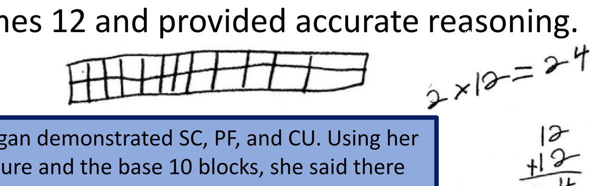
Lesson 7: Party Planning

We wanted to incorporate each of the previous standards and allow the students to use a strategy of their choice. Previously, students did not have to reason when to use each operation because the focus was either x or ÷. We decided to have students engage in such reasoning by solving multi-step/multi-operation word problems. We based many of our problems on examples in the Common Core State Standards Writing Team's (2011) progressions document.
- Cake: found the area of a sheet cake and determined the number of slices after being cut
-Goodie bags and balloons: solved multi-step/multi-operation story problems relating to the objects in the bags and the birthday balloons

Post-Assessment

All students exhibited Level 2 or Level 3 reasoning. Rather than counting 1-by-1, Piper used skip counting to determine the product. Megan and Trevor's automatic fact recall seemed to be accompanied more frequently by CU as they were able to identify the connection between the fact and their visual representation. All students demonstrated stronger AR. Elliott, Megan, and Tyler reflected on their work and identified a computational error they had made. Not only did Piper show stronger AR, but she also showed SC as she was able to explain her answers using the representations she created.

Array Model: All students accurately represented the brownie pan using a picture or manipulatives. They skip counted or multiplied 2 times 12 and provided accurate reasoning.



Megan demonstrated SC, PF, and CU. Using her picture and the base 10 blocks, she said there were 24 brownies in the pan.

Conceptualization of Division: Piper showed a much stronger conceptualization of division. Without being probed to do so, she used manipulatives to solve the school field trip problem.

Piper demonstrated SC with and CU of division. She knew 9 times something would yield 72. She counted out 72 plastic objects and evenly divided them into 9 piles (depicted above). From this, she reasoned 9 times 8 equals 72.

Reflection/Discussion

CCSSM 3.OA.7 (see above) was the most challenging standard for the students to attain because they had to know when to multiply/divide with accuracy and efficiency. In order to expose the students to various contexts in which multiplication/division occur, we based each lesson on a real-world context. They also had to become familiar with the operations' properties and various strategies. We helped the students develop these strategies by posing tasks in which it would be extremely inefficient to count 1-by-1 or skip count. As earlier work has indicated, Level 3 reasoning is the most difficult for students to attain because they need to have a CU of the operation and be able to pull from a repertoire of strategies to efficiently solve problems. Initially, Elliott, Megan, and Trevor only exhibited some Level 3 reasoning. As the summer progressed, the level of their reasoning did, as well. By the end of our sessions, they frequently used more sophisticated strategies that invoked the distributive and associative laws. This was especially apparent during lesson 8 when Elliott decomposed both factors, 9 and 21, to determine the area of a 9''x21'' sheet cake. Piper, the one student who did not initially exhibit Level 3 reasoning, demonstrated such reasoning during her post-assessment. Educators who aim to help their students attain 3.OA.7 could keep in mind fluency is not measured by speed, but by one's ability to apply the properties and strategies they previously learned throughout the trajectory. 3.OA.7 is not an independent standard that is attained with flash cards and drills. It is a standard that is continually developed as students advance along the portion of the trajectory depicted above. It is also important for educators to help students relate multiplication to rectangular area (3.OA.B). While establishing this relationship, students can develop the strategy of decomposing a factor to determine the product, a strategy that can be used when working with any context, not just area. In conclusion, when helping students attain 3.OA.7, it is important to expose them to various real-world contexts and to pose tasks in which it would be inefficient to count 1-by-1 or skip count. This is likely to help students understand when to multiply/divide, develop strategies to do so efficiently, and engage in more sophisticated multiplicative reasoning.