A Path Toward Multiplicative Reasoning With Understanding

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Introduction

Students often have a surface level understanding of multiplication and division (Otto, 2011). They frequently learn these operations without understanding their meanings or applicability (Sellers, 2010).This lack of understanding impairs their abilities to transition from solely additive reasoning to multiplicative reasoning when appropriate (Switzer, 2010).

The purpose of this study was to create a lesson sequence to build students' conceptual understanding of multiplication and division in order to enhance their abilities to make multiplicative comparisons. If students are able to conceptually understand multiplication and division, then they can reason and make such comparisons more effectively (Switzer, 2010).

Research Question:
What are the components of a lesson sequence that fosters children's ability to make multiplicative comparisons?

Theoretical framework

The National Research Council (2001, p. 116) described five strands of mathematical proficiency that guided our research:

- **Conceptual understanding** refers to the integrated and functional grasp of mathematical ideas which enables students to learn new ideas by connecting those ideas to what they already know.
- **Procedural fluency** is defined as skill in carrying out procedures flexibly, accurately, efficiently and appropriately.
- **Strategic competence** is the ability to formulate, represent, and solve mathematical problems.
- **Adaptive reasoning** is the capacity for logical thought, reflection, explanation, and justification.
- **Productive disposition** is the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Each of these strands must be supported in order to attain mathematical proficiency. In our study, we drew upon resources capable of fostering students' development along the five strands (e.g., Otto, 2011; Sellers, 2010; Switzer, 2010). Our goal was to encourage mathematical reasoning and not just procedural thinking (Sellers, 2010). We set out to accomplish our instructional goals by incorporating conceptual tools such as partial products, partial quotients, and area models in lessons (Switzer, 2010).

Methodology – Participants and procedure

- **Grade Level:** Just completed 4th Grade going into 5th Grade **# Students**: 4 students: 2 girls, 2 boys
- **Student Participation Rate:** 100% for 3 of the students, one student missed one session
- **Duration:** pre-assessment interview (30 minutes), seven one-hour sessions and one post-assessment interview (30 minutes)
- **Pseudonyms of participants:** Billy, Jackie, Jordan, Marshall

Goals for Instruction from the Common Core State Standards

(National Governor's Association for Best Practices & Council of Chief State School Officers, 2010)

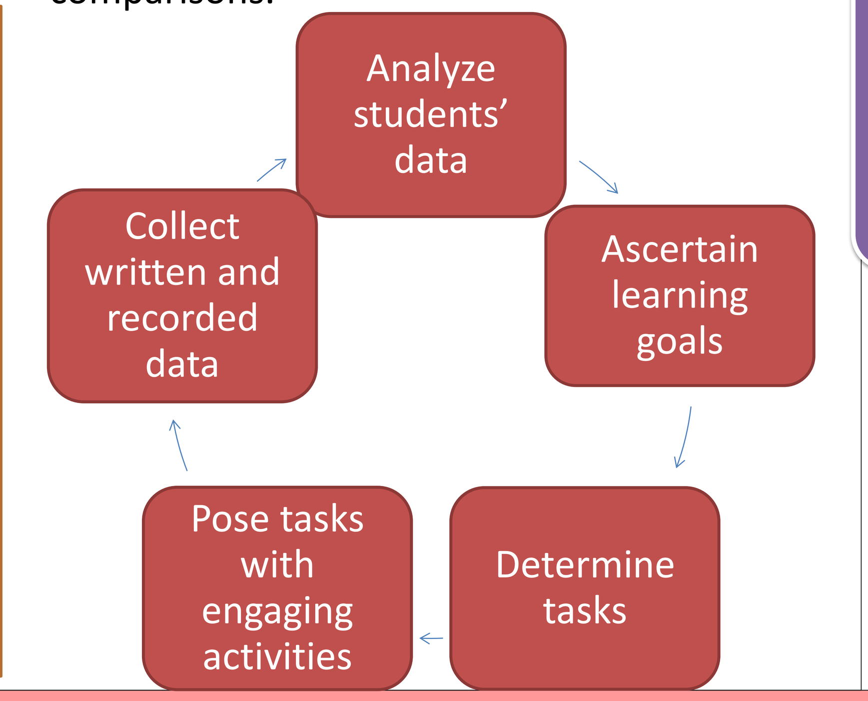
CCSS.MATH.CONTENT.4.OA.3- Solve word problems including interpreting remainders.

CCSS.MATH.CONTENT.4.MDA.3- Apply the area and perimeter formulas for rectangles in real world and mathematical problems.

CCSS.MATH.CONTENT.4.OA.1- Interpret a=b x c as a multiplicative comparison using times as many as times as much.

CCSS.MATH.CONTENT.4.OA.2- Solve word problems involving multiplicative comparisons.

Research Cycle:
Each week, we retained students' written work, video recorded our lessons, and transcribed them. We analyzed transcripts using the five strands of mathematical proficiency. We selected tasks for subsequent lessons that would help address students' mathematical proficiency weaknesses and build upon their strengths.



Methodology – Interview tasks

Pre and Post Interview: Key Tasks

A photographer has 191 photos of animals and 234 photos of plants. He wants to put all of the photos into photo books. Each page of the photo books hold 8 photos. What is the fewest number of pages he could use in the photo book?

For this task, we were looking to see which strategies the students would use. Some examples are the long division algorithm, creating a diagram, using manipulatives, etc.

Could you write an equation that represents the statement:

161 is 7 times as many as 23.

161 is 7 more than 154.

In this task, we were seeing if the students could formulate both additive and multiplicative equations. We were interested to see if students could distinguish between these two operations.

During a class trip to an apple farm, a group of students picked 2,436 apples. They packed them into 6 boxes to take to the local food bank. If each box held the same number of apple, how many apples were in each box?

We posed this problem to see how students would respond to a partitive (sharing) division situation, and if their strategies differed from those they used for other types of division.

References:

National Governor's Association for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/.

National Research Council. (2001*). Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Otto, A. D. (2011). *Developing essential understanding of multiplication and division for teaching mathematics in grades 3-5*. Reston, VA: National Council of Teachers of Mathematics.

Sellers, P. (2010). The trouble with long division. *Teaching Children Mathematics, 16*(9), 516-520.

Switzer, J. M. (2010). Bridging the math gap. *Mathematics Teaching in the Middle School, 15*(7), 400-405.

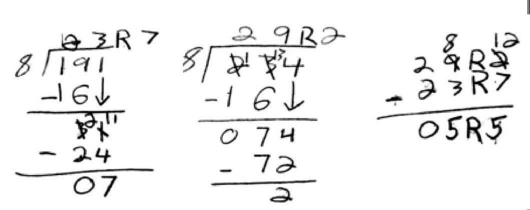
Empirical Teaching and Learning Trajectory:

Initial Assessment Results

Initial use of algorithms and procedures:

In the pre-interviews, most students showed strong procedural fluency with the exception of a few minor computational errors. Some struggled, however, to relate the results of executing a procedure back to the context of the original problem. Jordan's work, as shown in Figure 1, illustrates these general trends.

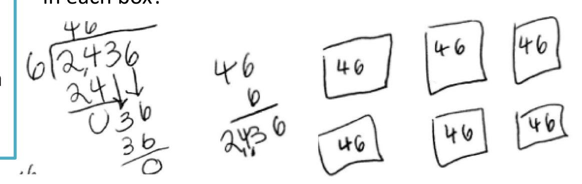
Figure 1: Jordan's work
A photographer has 191 photos of animals and 234 photos of plants. He wants to put all of the photos into photo books. Each page of the photo books holds 8 photos. What is the fewest number of pages he could use in the photo books?



Initial use of visual diagrams:

In the pre-assessment interviews, students tended to rely on the long division algorithm to solve problems even when they could not recall all of the steps in the algorithm. They were hesitant to use visual diagrams to solve problems. When they did use visual diagrams, they were employed to illustrate the answer obtained from performing an algorithm, as shown in Figure 2.

Figure 2: Billy's work
During a class trip to an apple farm, a group of students picked 2,436. They packed them into 6 boxes to take to the local food bank. If each box held the same number of apples, how many apples were in each box?



Initial attempts at multiplicative comparisons:

Jackie, Jordan and Billy struggled to create a multiplicative comparison from the statement (Figures 3, 4, and 5). They tried to compute the statement 161 is 7 times as many as 23, instead of creating a comparison. This showed a lack of strategic competence. Marshall, however, exhibited strategic competence by transforming the multiplicative statement into an equation.

Figure 3: Jordan's work

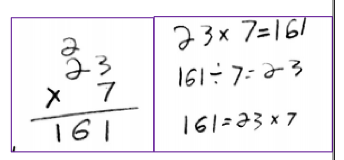


Figure 4: Jackie's Work

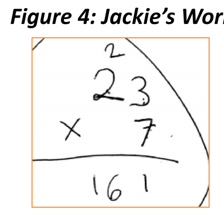
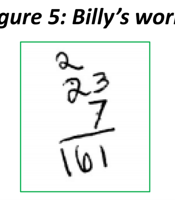


Figure 5: Billy's work



Partial quotients algorithm: Physical context connected to numbers

Our initial lessons incorporated problems designed to lead students to a partial quotients algorithm. Manipulatives such as unifix cubes were available to students during the lessons. We connected the physical chunking of manipulatives into groups with the idea of chunking numbers to create a concrete representation of a partial quotient algorithm.

As students engaged with the problems we posed, they were able to use partial quotients in different ways. For example, in the paperclip problem shown below, Billy repeatedly subtracted 200 to obtain a solution (Figure 7). Jackie was more efficient in her approach, subtracting larger chunks (Figure 6).

A paperclip company produced 1200 paperclips and wants to package them into bundles of 200. How many bundles can they sell?

Figure 6: Jordan's Work

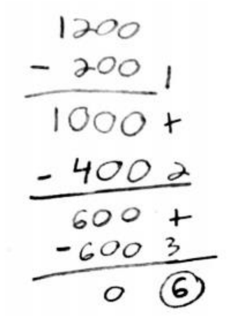
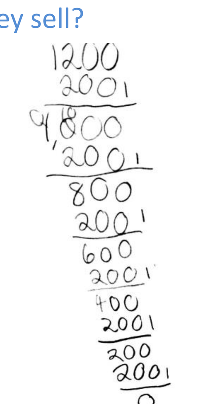


Figure 7: Billy's Work



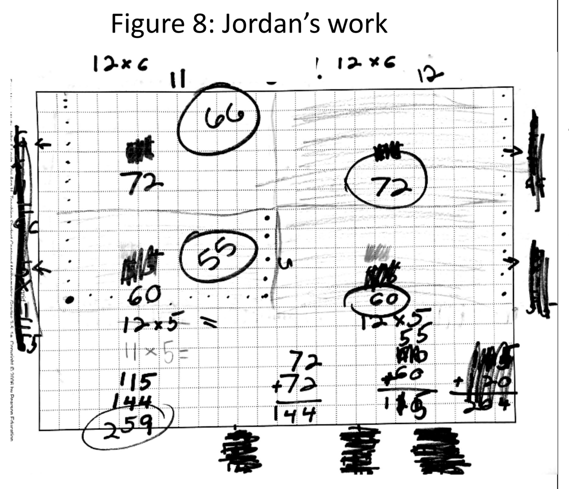
Jordan and Marshall, at times, clung to the standard algorithm for problems of this nature, even though they struggled to remember the steps of the long division algorithm or made errors that led them to the wrong answer. This motivated us to search for teaching strategies to help them employ more conceptual strategies.

Models: Area model multiplication/division

In an effort to foster students' conceptual understanding, we introduced area models. We prompted students to construct arrays using concrete materials. One such problem dealt with planting different crops in different sections of a garden. The problem and a student work sample (Figure 8) are shown below:

Harold, the gardener, has a 23 by 11 garden and he wants to plant carrots, beets, onions and corn in it. What are some ways in which Harold can design his garden? Use the starbursts to help Harold organize his garden.
Red starburst: Beets
Orange starburst: Carrots
Yellow starburst: Corn
Pink starburst: Onions

Figure 8: Jordan's work



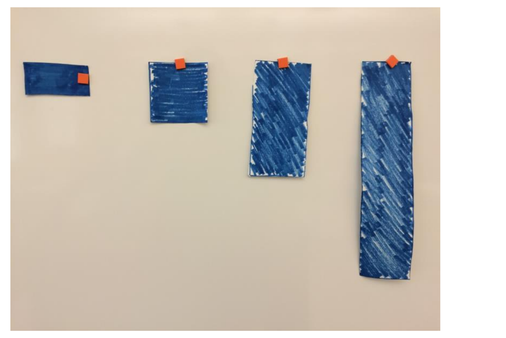
The work sample above demonstrates a typical student representation of the problem. Going forward, this type of representation gave students a way to think about products and quotients conceptually, in terms of area.

Multiplicative comparisons

Students' success with problems involving arrays suggested that they might be ready to engage in multiplicative comparisons. We presented the students four rectangles with the dimensions 5 by 10, 10 by 10, 20 by 10, and 40 by 10 (see Figure 9). The students immediately saw that the rectangle areas were doubling. When probed about their relationship from right to left, Jackie further stated that they were "half of one another." Marshall noticed that the 10 by 10 rectangle was one-fourth, in area, of the 40 by 10 rectangle.

We intentionally left key words out of the task statement to discourage the automatic use of procedures without connections. Therefore, the students had to understand the problem before they could generate their comparisons. Upon further probing, our students could successfully state how one rectangle's area and linear dimensions compared to the others. Their comparisons allowed them to see that as one linear dimension doubled, while the other remained constant, the area doubled. Marshall was able to reason that if the other linear dimension were to double as well, then the area itself would quadruple.

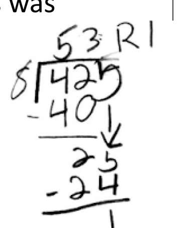
Figure 9:
These are the rectangles presented to the class in ascending order.



Post-Assessment Results

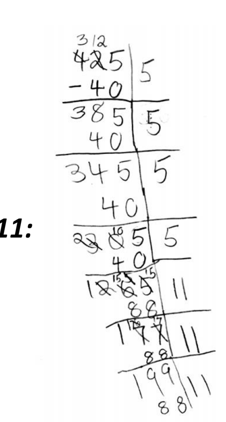
Jordan's ability to solve a problem involving multiple operations was fostered as shown by her final interview. Initially (see Figure 1), Jordan was not able to successfully compute the multistep item in our first interview session. After engaging in lessons that approached multiplication and division conceptually, she was able to think of and employ a strategy that allowed for successful completion(Figure 10). Jordan used the standard algorithm to successfully complete the multistep problem.

Figure 10:



Billy struggled with effectively computing the long division algorithm (see Figure 2) as he could not recall the "correct steps" during our initial interview. After introducing him to the partial quotients method, he was able to use this strategy efficiently to find the answer to the class trip problem shown in Figure 2. His post interview work is shown in Figure 11. This is Billy using the partial quotients method.

Figure 11:



In Jackie's initial interview her lack of confidence inhibited her from attempting problems. Throughout our time with her we worked on encouraging her to try new problems and to stop her negative self-talk. During the last sessions, she showed great confidence and actually jumped out of her seat to share the growth pattern she saw in the rectangle seen in Figure 9.

Marshall excelled in procedural fluency and had a hard time slowing himself down to really digest problems. Our lessons revolved around real world contexts that forced Marshall to think more deeply about the problem being asked. We encouraged him to make sense of the context and to connect the quantities to the context without immediately jumping into computations. It was evident in his post-interview that Marshall reflected upon answers.

Reflection and discussion:

Initially, the most difficult standards involved solving word problems with multiplicative comparisons (CCSS.MATH.CONTENT.4.OA.2) and those that invoked multiple operations and were considered "multistep" (CCSS.MATH.CONTENT.4.OA.3). The students struggled to formulate corresponding models and statements of multiplicative comparisons based on their context. They also struggled with choosing the appropriate operation and verbalized the need to find "key words" that their teachers had taught them. They jumped right to procedures without thinking about what the comparison really stated. We also found the students clung to the standard algorithms for multiplication and division, despite their failure to identify when errors had been made. In particular, their allegiance to the standard division algorithm hindered our exploration of multi-step word problems. Our research supports the learning progression recommendation that the standard algorithm for long division should be delayed until after the connection between computation and the context of the problem and the strategy being used is made. Our recommendation for teachers is to approach multiplication and division through problem solving situations (such as the garden problem next to Figure 8) that discourage the use of key words but focus on understanding. In particular, we found the partial quotients method for division to be a transparent method in that is computations could be easily tied to the physical context of the problem.